Separation of parameterized and delayed sources: application to multispectral data

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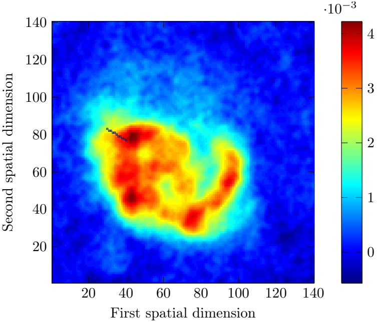
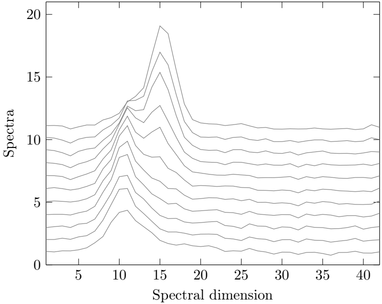
JATIA, Jan. 25, Strasbourg

Galaxy kinematics

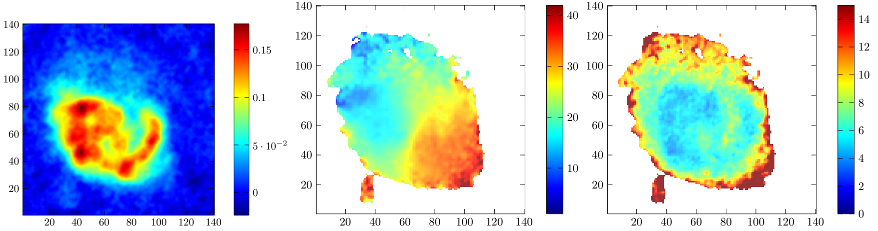
- The study of the intergalactic gas motion
- Data consist in multispectral images
- The peaks are shifted proportionally to the gas speed (Doppler effect)



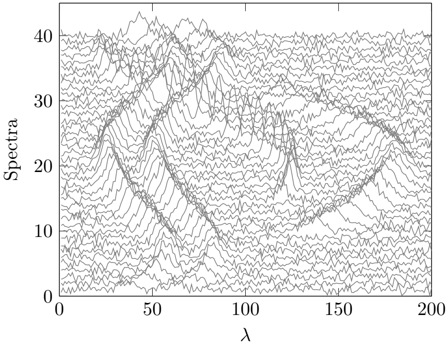


Moment method

$$M_0 = \sum_\lambda X(\lambda), \quad M_1 = \frac{\sum_\lambda X(\lambda)\lambda}{M_0}, \quad M_2 = \sqrt{\frac{\sum_\lambda (\lambda - M_1)^2 X(\lambda)}{M_0}},$$



Goal



Spectra
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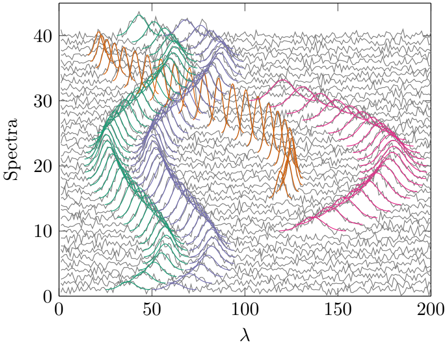

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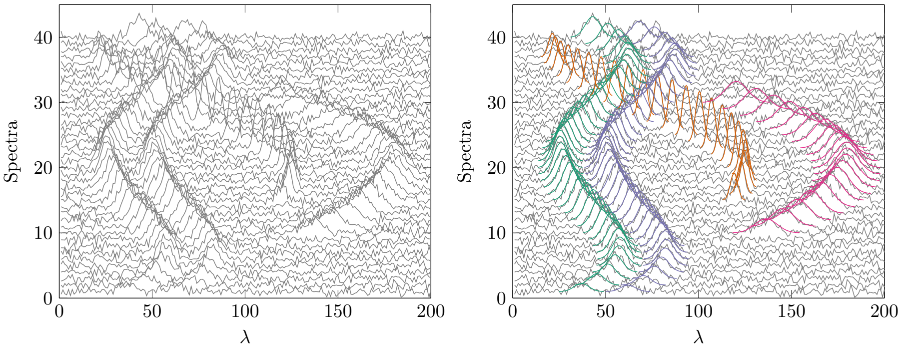
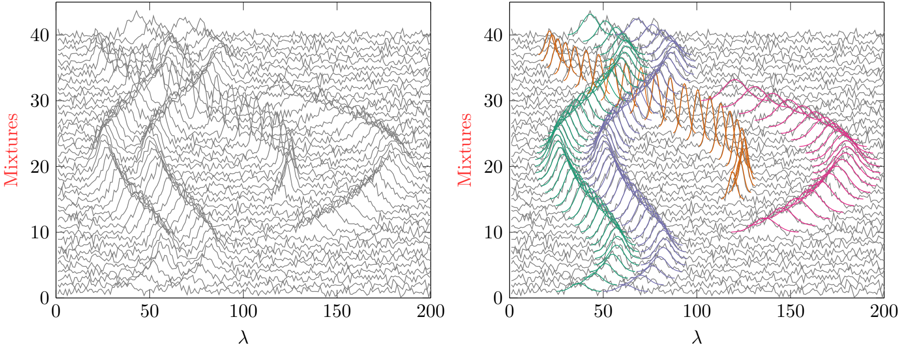


Challenges:

- Ill-posed inverse problem
- Similar and spectrally overlapping peaks
- Take into account the slow peak evolution

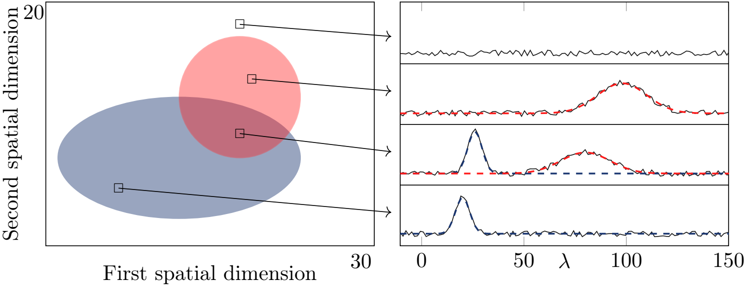
Source separation



$I = 40$ mixtures $J = 4$ sources

- Spectra i are assigned to mixtures
- Peaks j are assigned to sources
- Peak estimation and labeling are done simultaneously

Source separation



$I = 600$ mixtures $J = 2$ sources

- Spectra i are assigned to mixtures
- Peaks j are assigned to sources
- Peak estimation and labeling are done simultaneously

State of the art

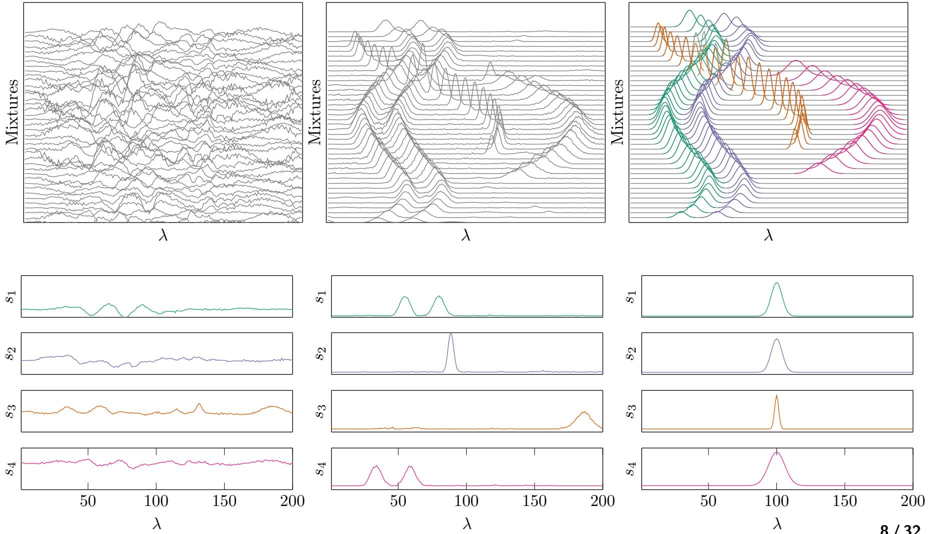
Blind delayed source separation

Bayesian method

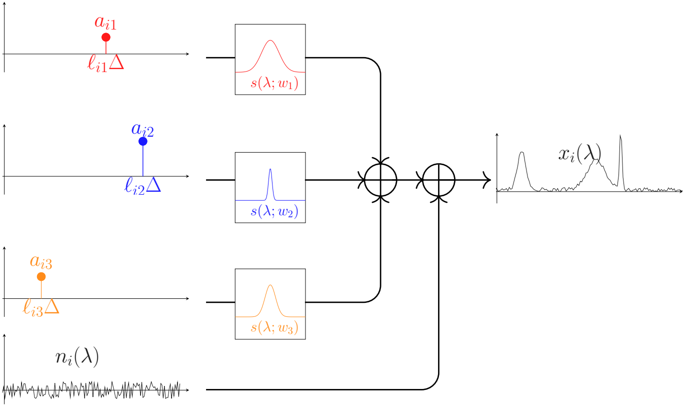
Nion *et al.*, *LVA/ICA*, 2010

Mörup *et al.*, *MLSP*, 2007

Mazet *et al.*, *Signal Process.*, 2015



Mixing model



$$\forall i, \quad x_i(\lambda) = \sum_{j=1}^{J} a_{ij} s(\lambda - \ell_{ij}\Delta; w_j) + n_i(\lambda)$$

$$\forall i, \quad \boldsymbol{x}_i = \sum_{j=1}^{J} a_{ij} \boldsymbol{s}[\ell_{ij}\Delta; w_j] + \boldsymbol{n}_i$$

Cost function

Estimate amplitudes $\boldsymbol{A}$, delays $\boldsymbol{L}$ and shape parameters $\boldsymbol{w}$ that minimize the total residual error:

$$E(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w}) = \qquad \left\| \boldsymbol{x}_i - \sum_{j=1}^{J} a_{ij} \boldsymbol{s}[\ell_{ij}\Delta; w_j] \right\|_2^2$$

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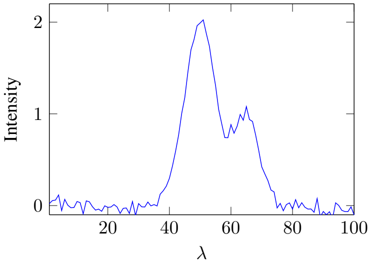
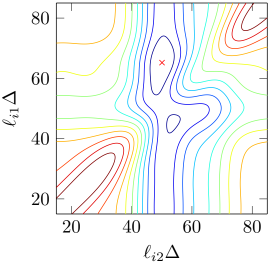
$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \dots & a_{1J} \\ \vdots & \ddots & \vdots \\ a_{I1} & \dots & a_{IJ} \end{bmatrix} \quad \boldsymbol{L} = \begin{bmatrix} \ell_{11} & \dots & \ell_{1J} \\ \vdots & \ddots & \vdots \\ \ell_{I1} & \dots & \ell_{IJ} \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} w_1 & \dots & w_J \end{bmatrix}$$

Cost function





Alternating Least Squares (ALS)

ALS Scheme:

Until convergence:

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1. minimize $E(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w})$ w.r.t. $\boldsymbol{w}$
   - Levenberg-Marquardt algorithm

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2. minimize $E(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w})$ w.r.t. $\boldsymbol{A}$ and $\boldsymbol{L}$
   - Sparse approximation (OMP-like implementation)

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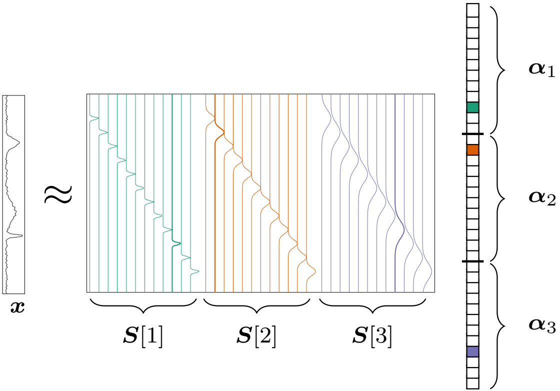
Separable problem w.r.t. to $\boldsymbol{A}$ and $\boldsymbol{L}$:

$$\min_{\boldsymbol{A}\geq\boldsymbol{0},\boldsymbol{L}} E\big(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w}\big) \Leftrightarrow \forall\ i, \quad \min_{\boldsymbol{A}_{i:}\geq\boldsymbol{0},\boldsymbol{L}_{i:}} \left\|\boldsymbol{x}_i - \sum_{j=1}^{J} a_{ij}\boldsymbol{s}[\ell_{ij}\Delta; w_j]\right\|_2^2$$

Sparse approximation



$$\boldsymbol{x} \approx \sum_{j=1}^{J} \boldsymbol{S}[j]\boldsymbol{\alpha}_j$$

ℓ_0 optimization problem:

$$\min_{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_J \geq \boldsymbol{0}} \left\| \boldsymbol{x} - \sum_{j=1}^{J} \boldsymbol{S}[j]\boldsymbol{\alpha}_j \right\|_2^2 \quad \text{s.t.} \quad \forall\ j,\ \|\boldsymbol{\alpha}_j\|_0 \leq 1$$

OMP-like implementation

OMP-like implementation for mixture i:

Till all the sources are selected:

$\left(\,\cdot\,\right)_+ = \max(\cdot, 0)$

OMP-like implementation

OMP-like implementation for mixture i:

Till all the sources are selected:

1. Find the most "positively" correlated delayed source to the residual
   - $\widehat{\ell}_{ij} \leftarrow \underset{\ell}{\operatorname{argmax}} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell\Delta; w_j] \right)_+$

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3. Estimate the amplitudes in the non-negative least-squares sense

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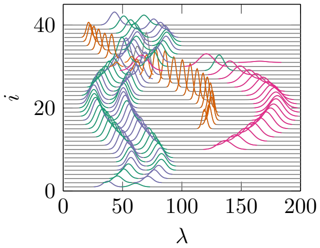
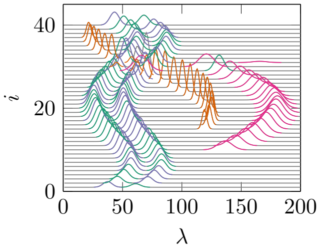
2. Find the associated source to exclude it from next selections
3. Estimate the amplitudes in the non-negative least-squares sense
4. Remove the selected delayed source from the residual

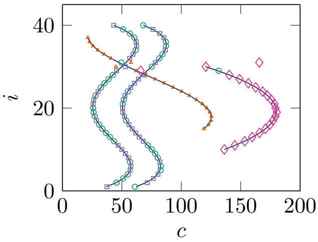
$(\,\cdot\,)_+ = \max(\cdot, 0)$

Results



Results





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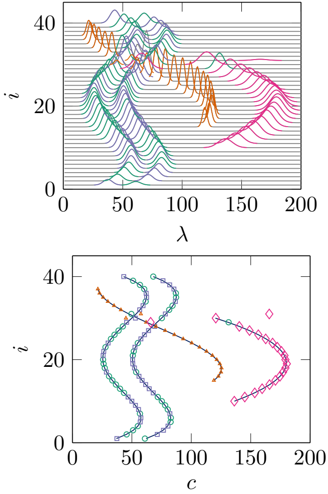

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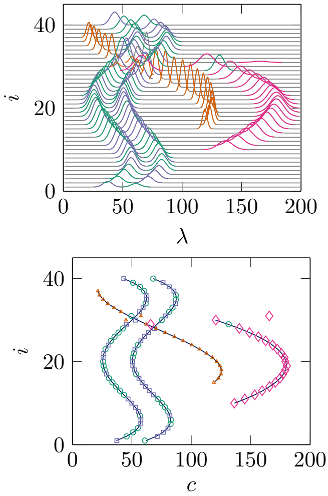

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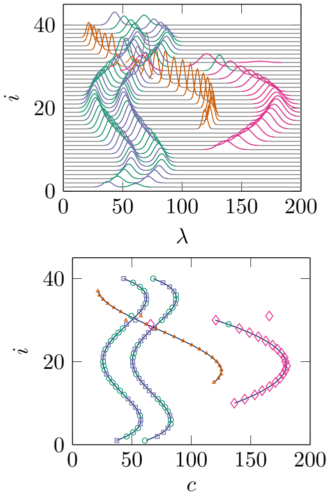

- Comp. time: 1.9 s
- Ambiguity in the separation of sources with very similar shapes

Regularized criterion

Motivations:

- New discriminating factor to separate highly correlated sources
- Neighbor acquisitions are similar

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$$F(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w}) = \underbrace{E(\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{w})}_{\text{Data-fit}} + \tau \underbrace{\Delta^2 \sum_{(i,i') \in \mathcal{G}} \sum_{j=1}^{J} (\ell_{ij} - \ell_{i'j})^2}_{\text{Delay regularization}}$$

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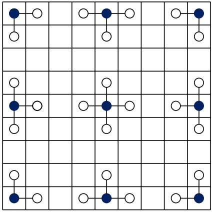
$\mathcal{G}$: set of cliques



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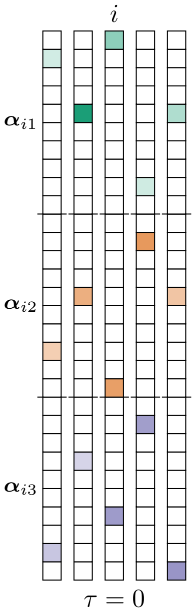
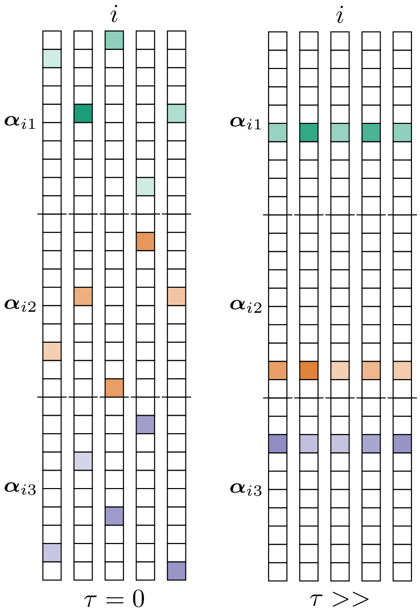
   - Joint sparse approximation

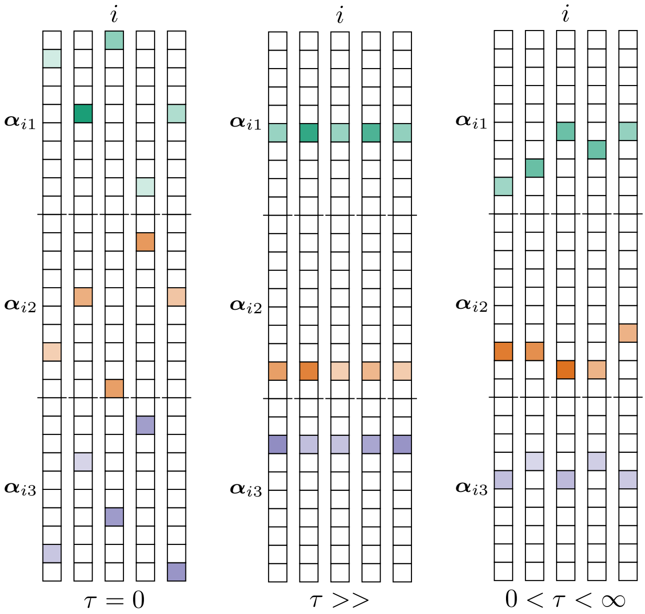
Neighbor sparse solutions



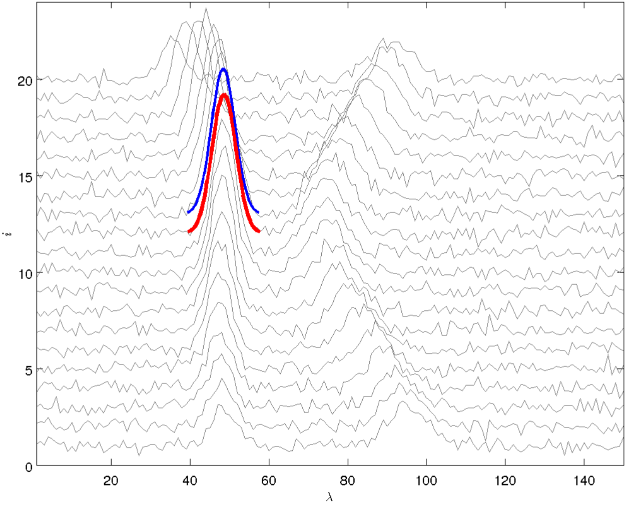
Neighbor sparse solutions



Neighbor sparse solutions



Joint NN-OMP



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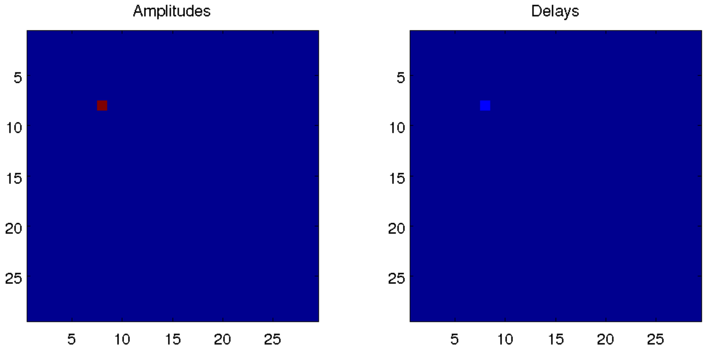

Amplitudes
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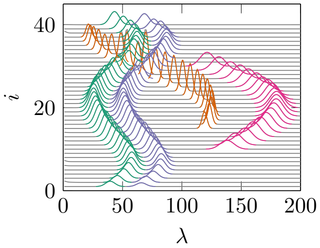
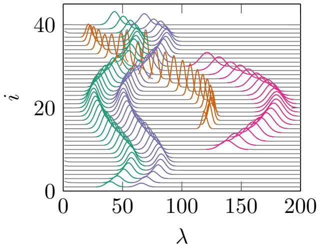
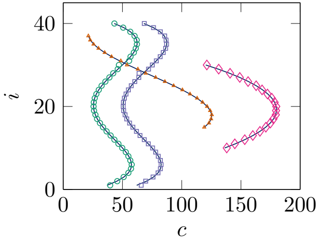
Results



Results





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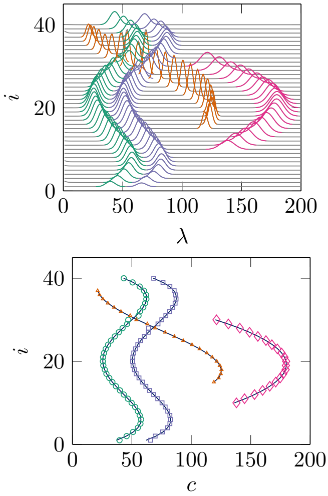

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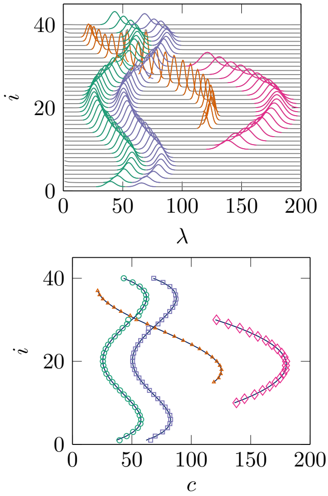
Results



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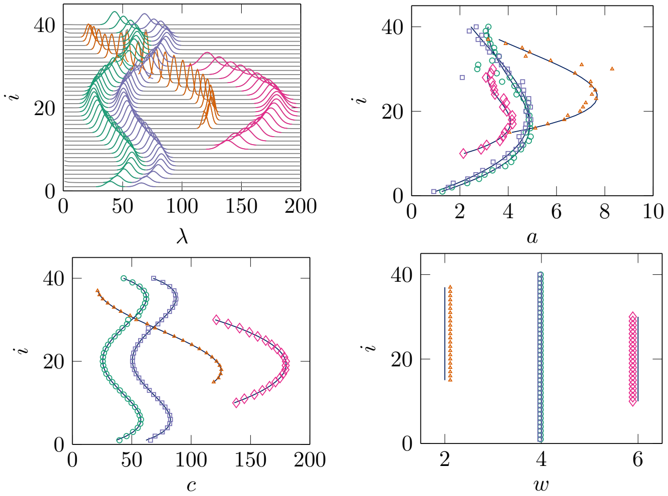
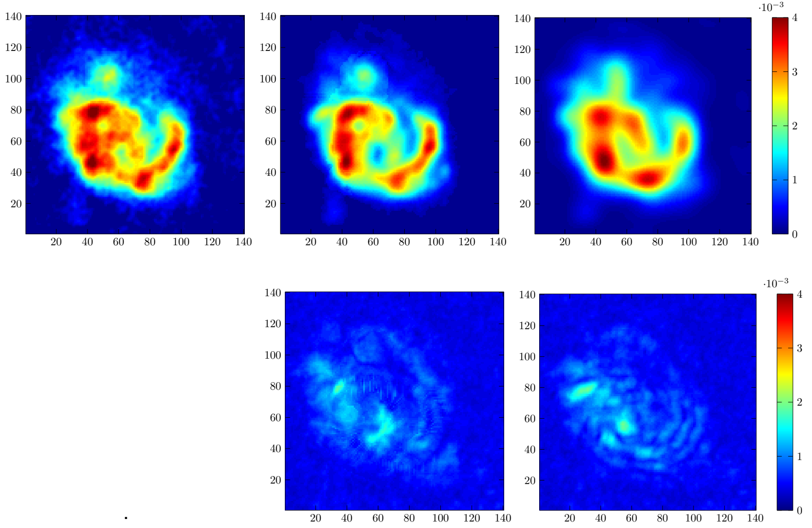


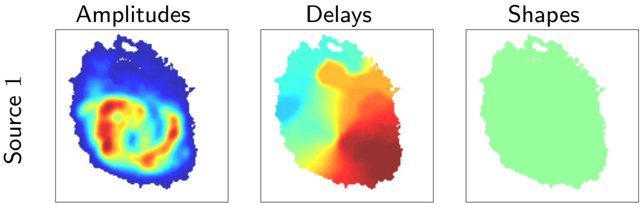
- Comp. time: 1 s

Galaxy NGC-4254

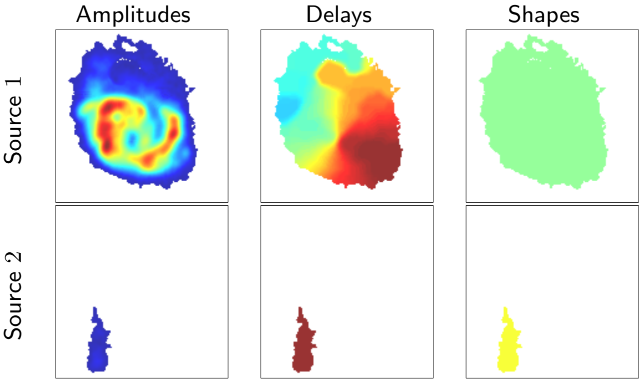
Data | Regularized method (5 min) | B-spline method (6 h)



Galaxy NGC-4254



Galaxy NGC-4254



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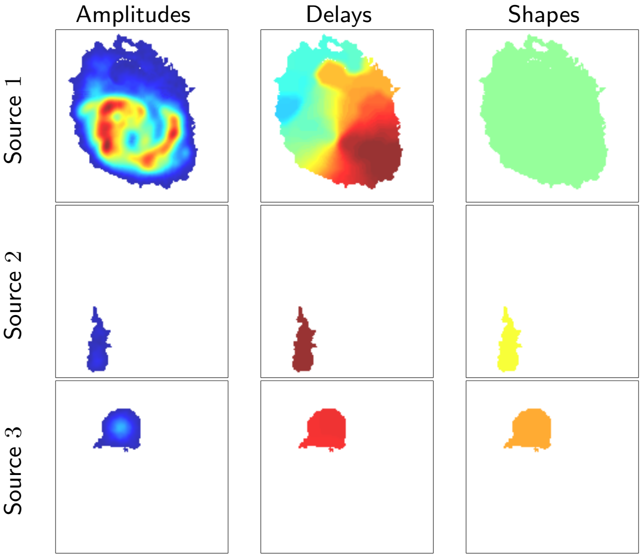
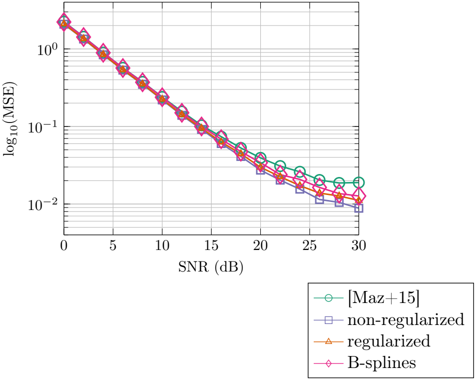


Method comparison

SNR ($0 \to 30$ dB), 1000 datasets: $I = 30$ mixtures, $J = 3$ sources, $N = 200$ samples



$$\mathrm{MSE} = \frac{1}{N \cdot I} \sum_{i=1}^{I} \left\| \boldsymbol{x}_i - \sum_{j=1}^{J} \widehat{a}_{ij} \, \boldsymbol{s}[\widehat{c}_{ij}; \widehat{w}_{ij}] \right\|_2^2$$

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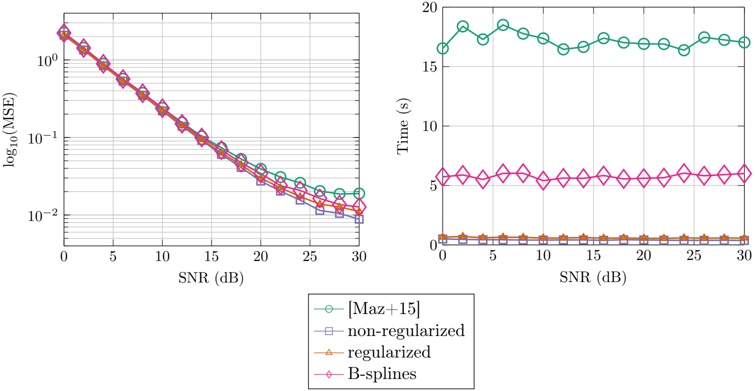

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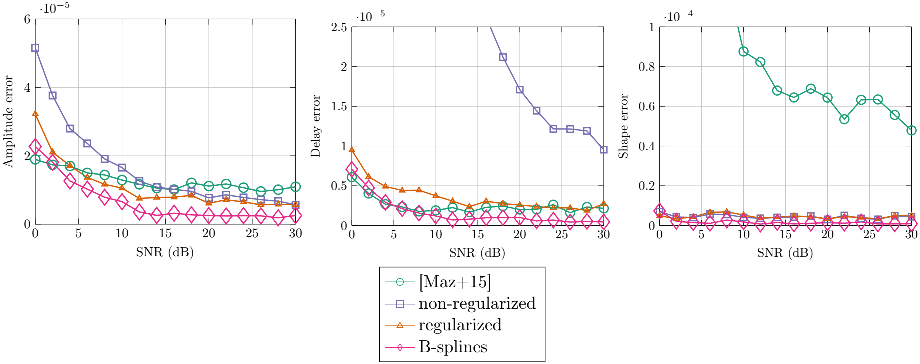

$$\text{error} = \frac{\|\mathbf{\Theta}^* - \widehat{\mathbf{\Theta}}\|_F^2}{\|\mathbf{\Theta}^*\|_F^2}$$

Conclusion

- Processing of multispectral data

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- Goals: estimate the peak characteristics (amplitudes, positions, shapes) and track their evolution

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- Goals: estimate the peak characteristics (amplitudes, positions, shapes) and track their evolution
- Parameterized source separation framework to achieve both goals simultaneously

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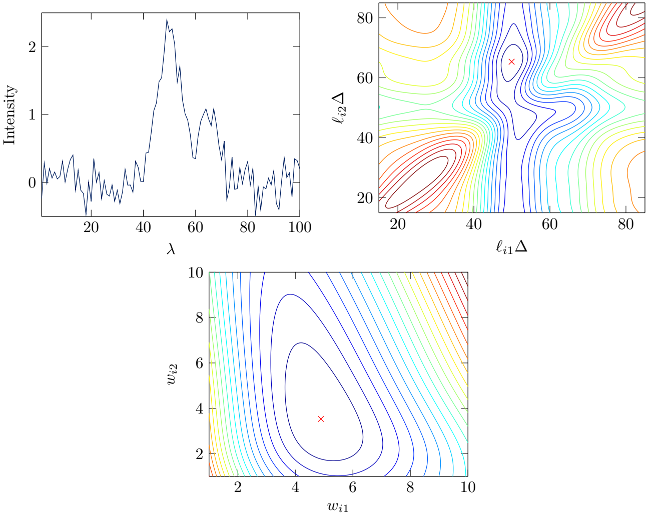
- Slow delay evolution enforcement by introducing a regularization and a joint sparse approximation algorithm
- Application to real data (galaxy kinematics)
- The proposed methods are more adapted to the problem than literature delayed source separation approach and much faster than a Bayesian method

Publications

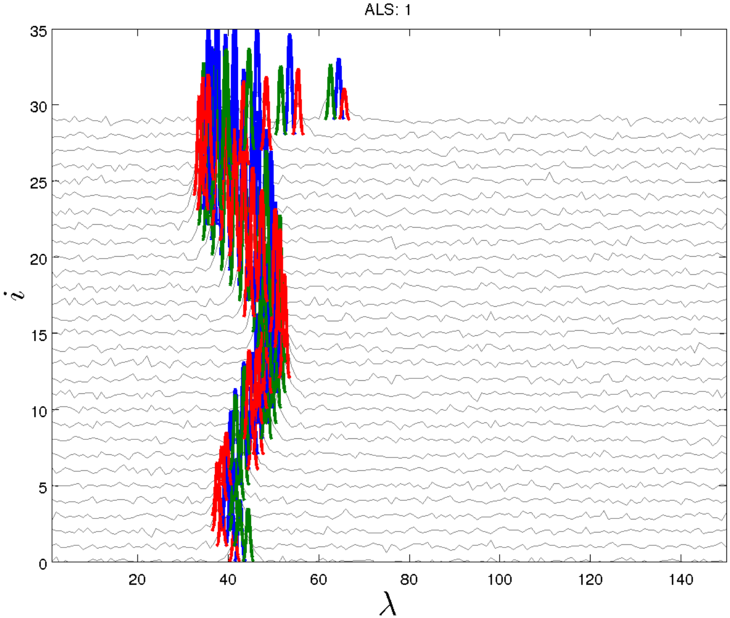
- H. Mortada, V. Mazet, C. Soussen, C. Collet, and L. Poisson. “Parameterized source separation for delayed spectroscopic signals”. Signal Processing, vol. 158, pp. 48-60, may 2019.
- H. Mortada, V. Mazet, C. Soussen, and C. Collet. “Separation of delayed parameterized sources”. In: Proceedings of the European Signal Processing Conference, EUSIPCO, pp. 1080-1084, 2017.
- H. Mortada, V. Mazet, C. Soussen, and C. Collet. “Spectroscopic decomposition of astronomical multispectral images using B-splines”. In: Workshop on Hyperspectral Images and Signal Processing, WHISPERS, 2018.
- H. Mortada, V. Mazet, C. Soussen, and C. Collet. “Séparation de sources retardées et paramétriques”. In: GRETSI, 2017.


THANKS!

Cost function



w initialization



Joint OMP with slow delay evolution

Joint OMP with slow delay evolution

- For $k = 1 \nearrow J$
  - **for** $j \in \{1, \ldots, J\} \backslash \mathcal{J}$ **do**
    $\widetilde{\boldsymbol{L}}_{:j} \leftarrow \underset{\boldsymbol{L}_{:j}}{\operatorname{argmax}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell_{ij}\Delta; \widehat{w}_j] \right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} (\ell_{ij} - \ell_{i'j})^2$ **end**
  - $\widehat{j} \leftarrow \underset{j \notin \mathcal{J}}{\operatorname{argmax}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\widetilde{\ell}_{ij}\Delta; \widehat{w}_j] \right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} (\widetilde{\ell}_{ij} - \widetilde{\ell}_{i'j})^2$
  - $\mathcal{J} \leftarrow \mathcal{J} \cup \{ \widehat{j} \}$
  - $\widehat{\boldsymbol{L}}_{:\widehat{j}} \leftarrow \widetilde{\boldsymbol{L}}_{:\widehat{j}}$
  - $\widehat{\boldsymbol{A}} \leftarrow \underset{\boldsymbol{A}}{\operatorname{argmin}}\, F(\boldsymbol{A}, \widehat{\boldsymbol{L}}, \widehat{\boldsymbol{w}})$ s.t. $\{\boldsymbol{A}_{:\mathcal{J}} \geq \boldsymbol{0}, \boldsymbol{A}_{:\overline{\mathcal{J}}} = \boldsymbol{0}\}$
  - **for** $i = 1 \nearrow I$ **do** $\quad \boldsymbol{r}_i \leftarrow \boldsymbol{x}_i - \sum_{j\in\mathcal{J}} \widehat{a}_{ij} \boldsymbol{s}[\widehat{\ell}_{ij}\Delta; \widehat{w}_j]$ **end**
- end

ICM strategy

$$\widetilde{\boldsymbol{L}}_{:j} \leftarrow \underset{\boldsymbol{L}_{:j}}{\operatorname{argmax}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell_{ij}\Delta; \widehat{w}_j]\right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} (\ell_{ij} - \ell_{i'j})^2$$

ICM strategy

- Sequentially sweep $i \in \{1, \ldots, I\}$
  - $\widetilde{\ell}_{ij} \leftarrow \underset{\ell}{\operatorname{argmax}} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell\Delta; \widehat{w}_j]\right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} \left(\ell - \widetilde{\ell}_{i'j}\right)^2$
- end

B-spline initialization

B-spline iterative initialization

- For $k = 1 \nearrow J$
  - Solve:
    - $$(\widehat{\boldsymbol{\Phi}}_{:,1:k}, \widehat{\boldsymbol{\Sigma}}_{:,1:k}, \widehat{\boldsymbol{\Omega}}_{:,1:k}) \leftarrow \underset{\boldsymbol{\Phi}_{:,1:k}, \boldsymbol{\Sigma}_{:,1:k}, \boldsymbol{\Omega}_{:,1:k}}{\operatorname{argmin}} \boldsymbol{L}(\boldsymbol{\Phi}, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$$
    - using a SQP algorithm, initialized as follows:
    - $$\boldsymbol{\Phi}^0_{:,1:k} = \left[\widehat{\boldsymbol{\Phi}}_{:,1:k-1} \quad \boldsymbol{\phi}^0_k\right] \text{ with } \boldsymbol{\phi}^0_k \sim \mathcal{U}(0, +\infty)$$

      $$\boldsymbol{\Omega}^0_{:,1:k} = \left[\widehat{\boldsymbol{\Omega}}_{:,1:k-1} \quad \boldsymbol{\omega}^0_k\right] \text{ with } \boldsymbol{\omega}^0_k \sim \mathcal{U}(w_{\text{min}}, w_{\text{max}})$$

      $$\boldsymbol{\Sigma}^0_{:,1:k} = \left[\widehat{\boldsymbol{\Sigma}}_{:,1:k-1} \quad \boldsymbol{\sigma}^0_k\right] \text{ with } \boldsymbol{\sigma}^0_k = \lambda_{\text{max}} \mathbf{1}_M$$

      λ_{max} is the wavelength with the highest energy in the average mixture:
- end

NN-OMP vs NN-FPG

- SNR: $5, 15, 25$ dB, 200 datasets with $I = 1$, $J = 3$, $N = 200$
- TP $=$ Card$(\mathcal{S}^* \cap \widehat{\mathcal{S}})$, FP $=$ Card$(\widehat{\mathcal{S}})$ $-$ TP

SNR		NN-OMP	NN-FPG
5 dB	MSE	$47 \cdot 10^{-3}$	$44 \cdot 10^{-3}$
	Sparsity	4.3	14.1
	TP	0.5	0.5
	FP	3.8	13.6
15 dB	MSE	$4 \cdot 10^{-3}$	$4 \cdot 10^{-3}$
	Sparsity	5.2	12.2
	TP	1.3	0.9
	FP	3.9	11.3
25 dB	MSE	$0.5 \cdot 10^{-3}$	$0.4 \cdot 10^{-3}$
	Sparsity	6.1	12.7
	TP	2.0	1.2
	FP	4.0	11.5

Slow varying shape extension

$$\boldsymbol{x}_i = \sum_{j=1}^{J} a_{ij}\boldsymbol{s}[\ell_{ij}\Delta; p_{ij}\Theta] + \boldsymbol{n}_i$$

$$G\big(\boldsymbol{A},\boldsymbol{L},\boldsymbol{P}\big) = \underbrace{F\big(\boldsymbol{A},\boldsymbol{L},\boldsymbol{P}\big)}_{\text{Data-fit and delay regularization}} + \mu \underbrace{\Theta^2 \sum_{(i,i')\in\mathcal{G}} \sum_{j=1}^{J} (p_{ij} - p_{i'j})^2}_{\text{Shape regularization}}$$

Slow varying shape extension

Joint OMP with slow delay evolution

- For $k = 1 \nearrow J$
  - **for** $j \in \{1, \ldots, J\} \backslash \mathcal{J}$ **do**
    $(\widetilde{\boldsymbol{L}}_{:j}, \widetilde{\boldsymbol{P}}_{:j}) \leftarrow \underset{\boldsymbol{L}_{:j}}{\operatorname{argmax}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell_{ij}\Delta; p_{ij}\Theta] \right)_+^2 - \tau \Delta^2 \sum_{(i,i') \in \mathcal{G}} (\ell_{ij} - \ell_{i'j})^2 - \mu \Theta^2 \sum_{(i,i') \in \mathcal{G}} (p_{ij} - p_{i'j})^2$ **end**
  - $\widehat{j} \leftarrow \underset{j \notin \mathcal{J}}{\operatorname{argmax}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\widetilde{\ell}_{ij}\Delta; \widetilde{p}_{ij}\Theta] \right)_+^2 - \tau \Delta^2 \sum_{(i,i') \in \mathcal{G}} (\widetilde{\ell}_{ij} - \widetilde{\ell}_{i'j})^2 - \mu \Theta^2 \sum_{(i,i') \in \mathcal{G}} (p_{ij} - p_{i'j})^2$
  - $\mathcal{J} \leftarrow \mathcal{J} \cup \{ \widehat{j} \}$
  - $(\widehat{\boldsymbol{L}}_{:\widehat{j}}, \widehat{\boldsymbol{P}}_{:\widehat{j}}) \leftarrow (\widetilde{\boldsymbol{L}}_{:\widehat{j}}, \widetilde{\boldsymbol{P}}_{:\widehat{j}})$
  - $\widehat{\boldsymbol{A}} \leftarrow \underset{\boldsymbol{A}}{\operatorname{argmin}}\, G(\boldsymbol{A}, \widehat{\boldsymbol{L}}, \widehat{\boldsymbol{P}})$ s.t. $\{\boldsymbol{A}_{:\mathcal{J}} \geq \boldsymbol{0}, \boldsymbol{A}_{:\overline{\mathcal{J}}} = \boldsymbol{0}\}$
  - **for** $i = 1 \nearrow I$ **do** $\boldsymbol{r}_i \leftarrow \boldsymbol{x}_i - \sum_{j \in \mathcal{J}} \widehat{a}_{ij} \boldsymbol{s}[\widehat{\ell}_{ij}\Delta; \widehat{p}_{ij}\Theta]$ **end**
- end

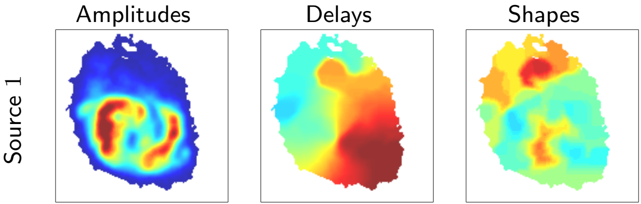
Slow varying shape extension

ICM strategy

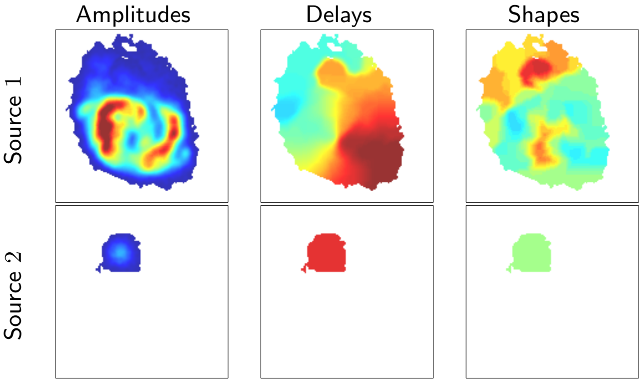
$$(\widetilde{\boldsymbol{L}}_{:j}, \widetilde{\boldsymbol{P}}_{:j}) \leftarrow \operatorname*{argmax}_{\boldsymbol{L}_{:j}, \boldsymbol{P}_{:j}} \sum_{i=1}^{I} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell_{ij}\Delta; p_{ij}\Theta]\right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} (\ell_{ij} - \ell_{i'j})^2 - \mu\Theta^2 \sum_{(i,i')\in\mathcal{G}} (p_{ij} - p_{i'j})^2$$

- Sequentially sweep $i \in \{1, \ldots, I\}$
  - $(\widetilde{\ell}_{ij}, \widetilde{p}_{ij}) \leftarrow \operatorname*{argmax}_{\ell, p} \left(\boldsymbol{r}_i^T \boldsymbol{s}[\ell\Delta; p\Theta]\right)_+^2 - \tau\Delta^2 \sum_{(i,i')\in\mathcal{G}} \left(\ell - \widetilde{\ell}_{i'j}\right)^2 - \mu\Theta^2 \sum_{(i,i')\in\mathcal{G}} (p - \widetilde{p}_{i'j})^2$
- end

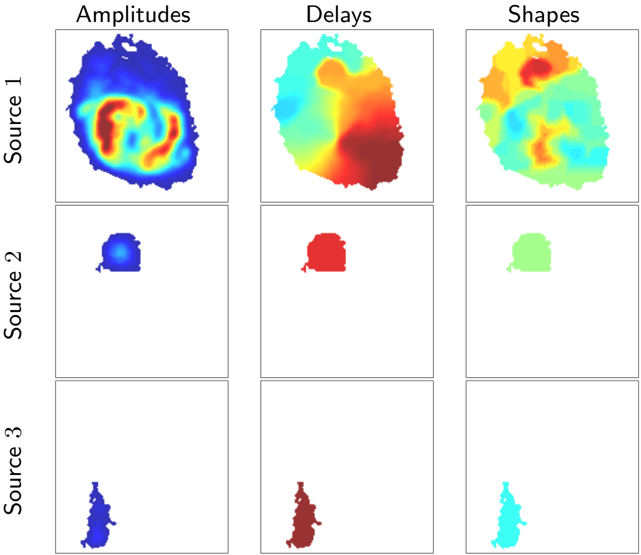
Galaxy NGC-4254



Galaxy NGC-4254



Galaxy NGC-4254



Solver comparison (B-spline method)

100 datasets, SNR $= 15$ dB, $I = 40$ mixtures, $J = 3$ sources and $N = 150$ samples.

	trust-region-reflective	active-set	interior points	SQP
MSE	1.1×10^{-1}	1.1×10^{-1}	1.2×10^{-1}	1.0×10^{-1}
Time (s)	4.12	4.88	3.5	4.7